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Author(s)	Vitturi, Marco; Wright, James
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MULTIPARAMETER SINGULAR INTEGRALS ON THE HEISENBERG GROUP: UNIFORM ESTIMATES

MARCO VITTURI AND JAMES WRIGHT

Abstract. We consider a class of multiparameter singular Radon integral operators on the Heisenberg group $\mathbb{H}^1$ where the underlying submanifold is the graph of a polynomial. A remarkable difference with the euclidean case, where Heisenberg convolution is replaced by euclidean convolution, is that the operators on the Heisenberg group are always L^2 bounded. This is not the case in the euclidean setting where L^2 boundedness depends on the polynomial defining the underlying surface. Here we uncover some new, interesting phenomena. For example, although the Heisenberg group operators are always L^2 bounded, the bounds are *not* uniform in the coefficients of polynomials with fixed degree. When we ask for which polynomials uniform L^2 bounds hold, we arrive at the *same* class where uniform bounds hold in the euclidean case.

1. Introduction

For the general theory of singular Radon transforms

$$H_{\gamma,K}f(x) \;=\; \psi(x)\int_{\mathbb{R}^k} f(\gamma(x,t))K(t)\,dt$$

where K is a singular kernel and $\gamma : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a smooth map (ψ an appropriate cut-off function), the case of translation-invariant polynomial mappings $\gamma(x,t) = x \cdot \Phi(t)$ has served as a model problem. Here $\Phi(t) = (P_1(t), \ldots, P_n(t))$ with polynomial components $P_j \in \mathbb{R}[X_1, \ldots, X_k]$ and the translation $\cdot$ arises from a nilpotent Lie group structure on $\mathbb{R}^n$. See [4] where the analysis of general singular Radon transforms $H_{\gamma,K}$ is effectively reduced to the case $\gamma(x,t) = x \cdot \Phi(t)$ described above in the one-parameter setting; that is, when K is a classical Calderón-Zygmund kernel satisfying $|\partial^\alpha K(t)| \lesssim |t|^{-k-|\alpha|}$ for all α and with appropriate cancellation conditions imposed.

In the euclidean translation-invariant case $\gamma(x,t) = x + \Phi(t)$ where Φ is a polynomial, one consequence of the powerful technique of *lifting* the problem to higher dimensions where Φ becomes a monomial map (see [16]) is that the proof of boundedness of the operator $H_{\gamma,K} = H_\Phi$ in fact proves the stronger statement that the bound can be taken to be independent of the polynomial Φ, once the degree of Φ is fixed. This is especially the case in the one-parameter setting; see [14] where the lifting technique is developed systematically and consequences are explored.

For multiparameter singular kernels K (see Section 2 for a precise definition), the operators $H_{\gamma,K}$ may or may not be L^2 bounded and matters depend on cancellation conditions which arise through a subtle interaction between the mapping γ and the kernel K. In the euclidean translation-invariant setting, these cancellation

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conditions have been thoroughly investigated by Ricci and Stein in [12] (see [8] for earlier work). In particular Theorem 5.1 in [12] gives a sufficient condition (a cancellation condition involving both γ and K) which guarantees L^2 (even L^p) boundedness of the associated singular integral operators. One can then check in particular instances if these conditions are necessary.

For instance if $\gamma(x,t) = x + \Sigma(t)$ where $\Sigma(t) = (t, P(t))$ parametrises an $(n-1)$-dimensional polynomial surface with $P \in \mathbb{R}[X_1, \ldots, X_{n-1}]$, then the so-called multiple Hilbert transform along Σ, $H_{\gamma,\mathcal{K}} = \mathcal{H}_{P,\mathcal{K}}$ where[1]

$$\mathcal{H}_{P,\mathcal{K}} f(x,z) \;=\; p.v. \int_{\mathbb{R}^{n-1}} f(x-t, z-P(t))\, \mathcal{K}(t)\, dt,$$

is a typical example of a multiparameter singular Radon transform treated in [12] (see also [8]). Here the multiple Hilbert transform kernel $\mathcal{K}(t) = 1/t_1 \cdots t_{n-1}$ is the canonical multiparameter singular kernel. If $P(t) = \sum_\alpha c_\alpha t^\alpha$ is a real polynomial in $n-1$ variables, we define the *support of* P as $\Delta(P) = \{\alpha : c_\alpha \neq 0\}$. For any finite $\Delta \subset \mathbb{N}_0^{n-1}$, let $\mathcal{V}_\Delta$ denote the finite dimensional subspace of real polynomials P in n variables with $\Delta(P) \subseteq \Delta$.

The following theorem is essentially due to Ricci and Stein (see [12]).

Theorem. *Fix $\Delta \subset \mathbb{N}_0^{n-1}$. Then*

$$\sup_{P \in \mathcal{V}_\Delta} \|\mathcal{H}_{P,\mathcal{K}}\|_{L^2 \to L^2} \;<\; \infty \tag{1}$$

holds if and only if for every $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \Delta$, at least $n-2$ of the α_j's are even. Furthermore if α has 2 odd components, then for $P(t) = t^\alpha$, the individual operator $\mathcal{H}_{P,\mathcal{K}}$ is unbounded on L^2.

More precisely, the sufficiency part of this theorem follows from Theorem 5.1 in [12] via a standard lifting procedure (effectively freeing up the monomials of P) to an operator on a higher dimensional space of the form $\mathcal{H}_{Q,\mathcal{K}}$ where

$$Q(t) \;=\; (Q_\alpha(t))_{\alpha \in \Delta(P)} \quad \text{and each} \quad Q_\alpha(t) = t^\alpha.$$

One then checks that Q and $\mathcal{K}$ satisfy the cancellation condition of Theorem 5.1 in [12]. For the necessity it is a simple computation to check that if $P(t) = t^\alpha$ and α has 2 odd components, then $\mathcal{H}_{P,\mathcal{K}}$ is unbounded on L^2 (see [5]).

This result depends very much on the multiparameter singular kernel under consideration. If the multiple Hilbert transform kernel $\mathcal{K}$ is replaced by a different multiparameter singular kernel, the cancellation condition in Theorem 5.1 changes. See [22] where a *projected* version of $\mathcal{H}_{P,K}$ is considered for a fixed polynomial P but the multiparameter singular kernels K vary. A sharp result is established where uniformity in K is sought for a fixed polynomial P.

In a remarkable series of papers, the translation-invariant theory of Ricci and Stein was extended to the general non-translation-invariant setting by Stein and Street; [17, 21, 22, 18, 19] and [23]. In this work two conditions on γ are introduced, one is a curvature condition generalising the fundamental curvature condition in [4] and another is an algebraic condition which can be viewed as a strong cancellation condition. When these two conditions hold, L^2 bounds for $H_{\gamma,K}$ are deduced for

[1] When referring to the multiple Hilbert transform singular kernel $p.v.\, 1/t_1 \cdots t_{n-1}$, we will use the calligraphic notation $\mathcal{K}$ and respectively $\mathcal{H}$ for the associated operator to distinguish it from a general singular kernel K and associated operator H.

any multiparameter singular kernel K. These two conditions depend only on γ and so the cancellation condition is decoupled from the particular singular kernel under consideration. Hence the results obtained are valid for all multiparameter singular kernels. In many cases, when uniformity in K is sought, the algebraic or cancellation condition can be shown to be necessary. See [22] for details.

A fascinating example is given by $\gamma(\underline{x}, s, t) = \underline{x} \cdot \Sigma(s, t)$ where $\Sigma(s, t) = (s, t, P(s, t))$ parameterises the graph of a polynomial surface in $\mathbb{R}^3$ and $\cdot$ is the Heisenberg group $\mathbb{H}^1 \simeq \mathbb{R}^3$ multiplication; $(x, y, z) \cdot (u, v, w) = (x + u, y + v, z + w + 1/2(xv - yu))$. Interestingly, both conditions alluded to above are always satisfied in this case (see Section 3 for details) and hence in particular Street's L^2 theory shows that $H_{\gamma,\mathcal{K}}$ is bounded on L^2 for *any* real polynomial P. This is in sharp contrast to the above Ricci-Stein theorem which shows that in the euclidean translation-invariant case $\gamma(\underline{x}, s, t) = \underline{x} + \Sigma(s, t)$, L^2 boundedness depends on the particular polynomial $P(s, t)$. This extends to any real-analytic P and any multiparameter singular kernel K - see [22] and [18]. A formal statement of the result just described is as follows.

Theorem 1.1. *For any real polynomial $P(s, t)$ (or more generally any real-analytic P near the origin $(0, 0)$) and multiparameter singular kernel K, consider*

$$H_{P,K,\mathcal{R}} f(x, y, z) \;=\; \iint_{\mathcal{R}} f((x, y, z) \cdot (s, t, P(s, t))^{-1})\, K(s, t)\, dsdt \tag{2}$$

where $\mathcal{R} = \mathcal{R}_{a,b,c,d} = \{(s, t) : 0 < a \leq |s| \leq b, 0 < c \leq |t| \leq d\}$ is any "rectangle" but when P is real-analytic at the origin, we take b and d to be sufficiently small. Then $H_{P,K,\mathcal{R}}$ is bounded on $L^2(\mathbb{H}^1)$. Furthermore the bounds can be taken to be independent of the truncation $\mathcal{R}$.

The arguments developed in this paper will give an alternative proof of Theorem 1.1. See also Section 3 for an extension of Theorem 1.1.

Interestingly when we seek L^2 bounds, uniform with respect to the polynomial P as in the bound (1) appearing in the Ricci-Stein theorem, we come back to the euclidean conclusion of that theorem, as we will now state. For the double Hilbert transform kernel $\mathcal{K}(s, t) = 1/st$, define

$$H_{P,\mathcal{K},\mathcal{R}} f(x, y, z) := \iint_{\mathcal{R}} f((x, y, z) \cdot (s, t, P(s, t))^{-1}) \mathcal{K}(s, t)\, ds\, dt$$

where $\mathcal{R} = \mathcal{R}_{a,b,c,d} = \{(s, t) : 0 < a \leq |s| \leq b, 0 < c \leq |t| \leq d\}$ is a rectangle. Then we are able to show the following.

Theorem 1.2. *Fix $\Delta \subset \mathbb{N}_0^2$. Then*

$$\sup_{P \in \mathcal{V}_\Delta, \mathcal{R}} \|H_{P,\mathcal{K},\mathcal{R}}\|_{L^2(\mathbb{H}^1) \to L^2(\mathbb{H}^1)} \;<\; \infty \tag{3}$$

holds if and only if every $\alpha = (\alpha_1, \alpha_2) \in \Delta$ has at least one even component.

More generally, for $H_{P,K,\mathcal{R}}$ as in (2) where K is a general multiparameter singular kernel K, the uniformity in (3) is equivalent to the L^2 uniformity of a family of truncations[2] (with respect to the rectangles $\mathcal{R}$) of the singular Radon transform

$$R_{P,K}\, g(x, y) \;= p.v. \iint_{\mathbb{R}^2} g(x - t, y - P(s, t))\, K(s, t)\, ds\, dt.$$

[2]See Section 8 for a precise definition of the truncations.

The operator $R_{P,K}$ has not been specifically treated in the literature. It is a variant of the operators considered in [2] and it falls within the scope of Street's theory [23].

The rest of the paper is organised as follows. In Section 2 we briefly illustrate Street's L^2 theory in the biparameter case that will be relevant for our discussion. In Section 3 we present an extension of Theorem 1.1 (and Theorem 1.2 somewhat) to a larger class of graphs, for which we establish a characterisation of the $L^2(\mathbb{H}^1)$ boundedness. In Section 4 we use the group Fourier transform on $\mathbb{H}^1$ to reduce the problem to that of proving uniform L^2 boundedness for a class of integral operators acting on functions of one variable. The integral kernels of these operators will be given by a certain oscillatory integral expression involving P. In Section 5 we state and prove some oscillatory integral estimates of van der Corput type that will be helpful throughout. Section 6 contains the bulk of the proof, whose broad strategy consists in iteratively simplifying the phase of the aforementioned oscillatory integral kernels by stripping terms away from its Taylor expansion while using the estimates of Section 5 to keep the errors thus introduced under control. Once the phase has been simplified enough, the proofs of Theorems 1.2 and 3.1 are then concluded in the brief Sections 8 and 7, respectively. Finally, in Appendix A we prove a technical oscillatory integral inequality that appeared in Section 5.

Notation Uniform bounds for oscillatory integrals lie at the heart of this paper. Keeping track of constants and how they depend on the various parameters will be important for us. For the most part, constants C appearing in inequalities $A \leq CB$ between positive quantities A and B will be *absolute* or *uniform* in that they can be taken to be independent of the parameters of the underlying problem. We will use $A \lesssim B$ to denote $A \leq CB$ and $A \sim B$ to denote $C^{-1}B \leq A \leq CB$. If A is a general real or complex quantity, we write $A = O(B)$ to denote $|A| \leq CB$ and when we want to highlight a dependency on a parameter θ, we write $A = O_\theta(B)$ or $|A| \lesssim_\theta B$ to denote $|A| \leq C_\theta B$.

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2. The work of Street [22]

In [22], Street develops the L^2 theory for multiparameter singular Radon transforms

$$H_{\gamma,K}f(x) \;=\; \psi(x)\int_{\mathbb{R}^k} f(\gamma(x,t))\,K(t)\,dt$$

and introduces two key conditions on γ; a finite-type (curvature) condition and an algebraic (cancellation) condition. Here $\gamma : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ is a smooth map satisfying $\gamma(x,0) \equiv x$, ψ an appropriate cut-off function, and $K(t)$ is multiparameter singular kernel which is usually supported near the origin $t = 0$.

For our purposes it suffices to restrict our attention to the biparameter case $\mathbb{R}^k = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ and to *product kernels* K as introduced in [7], which underpins the theory of singular integrals with respect to flag kernels (however our analysis extends to treat the more general class of multiparameter singular kernels considered in [22]).

The notion of product kernel depends on the classical notion of Calderón-Zygmund kernels in one parameter; that is, a distribution K on $\mathbb{R}^k$ which coincides with a

smooth function away from the origin such that $|\partial^\alpha K(t)| \lesssim_\alpha |t|^{-k-|\alpha|}$ for all α and such that the quantities $\int K(t)\phi(Rt)dt$ are bounded, uniformly over all $R > 0$ and all smooth ϕ supported in the unit ball with $\|\phi\|_{C^1} \leq 1$ (such a ϕ is called a *normalised bump function* on $\mathbb{R}^k$).
A 2-parameter product kernel K is defined as follows. It is a distribution on $\mathbb{R}^k = \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$ which coincides with a C^∞ function K away from the coordinate subspaces $s = 0$, $t = 0$ and satisfies

1. (Differential inequalities) for every multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$, there is a constant C_α such that
$$|\partial_s^{\alpha_1}\partial_t^{\alpha_2} K(s,t)| \leq C_\alpha\, |s|^{-k_1-|\alpha_1|}|t|^{-k_2-|\alpha_2|}$$
away from the two coordinate subspaces, and
2. (Cancellation) for any normalised bump function ϕ on $\mathbb{R}^{k_1}$ and any $R > 0$, the distribution
$$K^1_{\phi,R}(t) \;=\; \int_{\mathbb{R}^{k_1}} K(s,t)\phi(Rs)\, ds$$
is a classical one-parameter Calderón-Zygmund kernel on $\mathbb{R}^{k_2}$ as described above. Similarly for $K^2_{\phi,R}(s) = \int K(s,t)\phi(Rt)dt$.

Important for our analysis is the following characterisation of product kernels; see Corollary 2.2.2 in [7]. For every smooth ϕ and $I = (j,k) \in \mathbb{Z}^2$, we set $\phi^{(I)}(s,t) := 2^{-j-k}\phi(2^{-j}s, 2^{-k}t)$.

Proposition 2.1. *A product kernel K can be written as*
$$K \;=\; \sum_{I\in\mathbb{Z}^2} \phi_I^{(I)} \tag{4}$$
(which is convergent in the sense of distributions) where each smooth ϕ_I is supported in $\{(s,t) : 1/2 \leq |s|, |t| \leq 2\}$, satisfies the cancellation conditions
$$\int \phi_I(s,t)\, ds \;\equiv\; 0 \quad\text{and}\quad \int \phi_I(s,t)\, dt \;\equiv\; 0 \tag{5}$$
for every t and s, and the sequence $\{\phi_I\}$ is bounded in C^k norm for every k.

The two key conditions on γ are easily formulated in the case where γ can be written as the exponential[3]
$$\gamma(x,(s,t)) \;=\; \exp\left(\sum s^p t^q X_{p,q}\right)(x) \tag{6}$$
of a *finite* sum of smooth vector fields $\{X_{p,q} = X_\alpha\}$. We assign to each X_α, where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$, the formal degree $d_\alpha = (|\alpha_1|, |\alpha_2|) \in \mathbb{N}\times\mathbb{N}$ and recursively we then define formal degrees for all iterated commutators such that if d_1 and $d_2 \in \mathbb{N}^2$ are the degrees of iterated commutators X_1 and X_2, respectively, then $[X_1, X_2]$ has degree $d_1 + d_2 \in \mathbb{N}\times\mathbb{N}$. Hence we view these vector fields together with their corresponding degree (X, d). Notice that it might be the case that one vector field has more than one degree; in this case we consider them to be distinct objects.

We separate the original vector fields $\{(X_\alpha, d_\alpha)\} = \mathcal{P}\cup\mathcal{N}$ into two types; the *pure ones* $(X_\alpha, d_\alpha) \in \mathcal{P}$ where $d_\alpha = (p, 0)$ or $d_\alpha = (0, q)$ and *non-pure ones* $(X_\alpha, d_\alpha) \in \mathcal{N}$ where $d_\alpha = (p, q)$ and both p and q are nonzero. The two key conditions on γ are

[3]The multiparameter exponential is to be interpreted as follows: for s, t given, define vector field $Y_{s,t}(x) = \sum s^p t^q X_{p,q}(x)$; then $\exp\left(\sum s^p t^q X_{p,q}\right)(x) := \exp(\tau Y_{s,t})\big|_{\tau=1}(x)$.

the following: there is a finite list $\{(X_1, d_1), \ldots, (X_N, d_N)\}$ of iterated commutators of pure vector fields, containing $\mathcal{P}$ itself and such that

1. (Finite-type condition) for all $\delta = (\delta_1, \delta_2) \in [0,1]^2$, we can write[4]
$$[\delta^{d_j} X_j, \delta^{d_k} X_k] \;=\; \sum_{\ell=1}^{N} c_{j,k}^{\ell,\delta} \delta^{d_\ell} X_\ell \tag{7}$$
where $c_{j,k}^{\ell,\delta} \in C^\infty$, uniformly in δ; and
2. (Algebraic condition) for $(Y, e) \in \mathcal{N}$ and every $\delta \in [0,1]^2$, we can write
$$\delta^e Y \;=\; \sum_{\ell=1}^{N} c_Y^{\ell,\delta} \delta^{d_\ell} X_\ell \tag{8}$$
where $c_Y^{\ell,\delta} \in C^\infty$, uniformly in δ.

Remark 1. Notice the two conditions imply that the involutive distribution generated by the collection $\{X_\alpha\}$ is finitely generated (as a C^∞-module). In the one-parameter case, this is essentially equivalent to the conditions above and the scaling factors in $\delta = \delta_1$ play essentially no active rôle. However, this is no longer necessarily true in the multiparameter case (see [22], Section 17.7) and the uniform behaviour in $\delta = (\delta_1, \delta_2)$ becomes crucial there.

The finite-type condition (7) is a generalisation of the curvature condition introduced in [4] in the one-parameter setting and the algebraic condition (8) allows us to *control* the troublesome non-pure vector fields $Y \in \mathcal{N}$ in terms of the pure ones, effectively transferring any needed cancellation down to the product kernel K. In this case, under these two conditions on γ, L^2 bounds for $H_{\gamma,K}$ can be derived for any product kernel K. In more general (non-finite, that is when γ is not exactly of type (6)) situations, the conditions (7) and (8) need to be modified. See [22] for details and in particular see section 3 of [22] for a discussion of the finite case discussed above.

3. Further results

The particular situation we are concerned with here is $\gamma(\underline{x}, (s,t)) = \underline{x} \cdot \Sigma(s,t)$ where the product $\cdot$ is the Heisenberg $\mathbb{H}^1$ group multiplication and $\Sigma(s,t) = (P_1(s,t), P_2(s,t), P_3(s,t))$ parametrises a surface in $\mathbb{H}^1$. Let $X = \partial_x - (y/2)\partial_z, Y = \partial_y + (x/2)\partial_z$ and $Z = \partial_z$ be the usual basis of left-invariant vector fields on $\mathbb{H}^1$ such that $[X, Y] = Z$. Then
$$\gamma(\underline{x}, (s,t)) \;=\; \underline{x} \cdot \Sigma(s,t) \;=\; \exp(P_1(s,t)X + P_2(s,t)Y + P_3(s,t)Z)(\underline{x}),$$
putting us in the above *finite* situation if each P_j is a polynomial. In this case the finite-type condition (7) is automatically satisfied. In turns out that when the P_j are (more generally) real-analytic, the appropriately modified finite-type condition (7) is still automatically satisfied; see [18].

In the case that $P_1(s,t) = s$ and $P_2(s,t) = t$, we see that $(X, (1,0))$ and $(Y, (0,1))$ lie in $\mathcal{P}$. Furthermore the only vector fields lying in $\mathcal{N}$ must be of the form (Z, d) where $d = (p,q)$ satisfies $pq \neq 0$ and the monomial $s^p t^q$ arises in the Taylor expansion of $P_3(s,t)$. Hence for any real-analytic P_3, every non-pure vector field in $\mathcal{N}$ can be

[4] Here $\delta^d = \delta^{(d_1,d_2)} := \delta_1^{d_1} \cdot \delta_2^{d_2}$.

controlled as described in (8) and so both conditions (7) and (8) are automatically satisfied when $\Sigma(s,t) = (s,t,P(s,t))$ is the graph of a real-analytic surface in $\mathbb{H}^1$. This is the background discussion for Theorem 1.1.

Now let us consider a slight variant; a surface parameterised by $\Sigma(s,t) = (s^{p_0}, t, P(s,t))$ where P is a general real-analytic function near $(0,0)$. As mentioned above, the corresponding finite-type condition (7) is automatically satisfied but now it is not necessarily the case that all non-pure vector fields $(Z, d') \in \mathcal{N}$ can be controlled by pure vector fields in the sense of (8). Recall that $d' = (p', q')$ where $p'q' \neq 0$ and $s^{p'}t^{q'}$ arises in the series expansion of $P(s,t) = \sum c_{p,q} s^p t^q$. Note that if $p' \geq p_0$, then we can control (Z, d') by (Z, d_0) where $d_0 = (p_0, 1)$ and (Z, d_0) arises as the commutator of the pure vector fields $(X, (p_0, 0))$ and $(Y, (0,1))$. Therefore the non-pure vector fields (Z, d') which cannot be controlled in the sense of (8) must necessarily satisfy $p' < p_0$ and so arise from a term in

$$P_{p_0}(s,t) \ := \ \sum_{p=0}^{p_0-1} \sum_{q \geq 1} c_{p,q} s^p t^q \ + \ \sum_{p \geq 1} c_{p,0} s^p.$$

When $p_0 = 1$, we have $P_{p_0}(s,t) = P(s,0)$ and so no $d' = (p', q')$ with $p'q' \neq 0$ satisfies $p' < p_0 = 1$, bringing us back to the case where all non-pure terms can be controlled by pure ones; that is, condition (8) is satisfied.
When $p_0 > 1$ the following result is thus new.

Theorem 3.1. *For any real-analytic $P(s,t)$ near the origin $(0,0)$ and multiparameter singular kernel K, consider*

$$H_{P,K,\mathcal{R}} f(x,y,z) \ = \ \iint\limits_{\mathcal{R}} f((x,y,z)\cdot(s^{p_0}, t, P(s,t))^{-1})\, K(s,t)\, ds\, dt$$

where $\mathcal{R} = \mathcal{R}_{a,b,c,d} = \{(s,t) : 0 < a \leq |s| \leq b, 0 < c \leq |t| \leq d\}$ lies in a small neighbourhood of the origin $(0,0)$. If $P_{p_0} \equiv 0$, then $H_{P,K,\mathcal{R}}$ is bounded on $L^2(\mathbb{H}^1)$.

In general, the $L^2(\mathbb{H}^1)$ boundedness of $H_{P,K,\mathcal{R}}$ is equivalent to the uniform $L^2(\mathbb{R}^2)$ boundedness of a family of truncations[5] *of the singular Radon transform*

$$R_{P_{p_0},K}\, g(x,y) \ = \ \iint g(x-t, y - P_{p_0}(s,t)) K(s,t)\, ds\, dt.$$

Furthermore when K is the double Hilbert transform kernel $\mathcal{K}(s,t) = p.v.\, 1/st$, then $H_{P,\mathcal{K},\mathcal{R}}$ is bounded on $L^2(\mathbb{H}^1)$ (uniformly in $\mathcal{R}$) if and only if every vertex (p,q) of the Newton polygon of P_{p_0} has the property that pq is even.

We recall that the Newton polygon of P_{p_0} is the convex hull of the quadrants $(p,q) + \mathbb{R}^2_+$ in $\mathbb{R}^2$ where $(p,q) \in \Delta(P_{p_0})$, the support of P_{p_0}. The rôle of the Newton polygon in the theory of multiparameter singular Radon transforms first appeared in [2].

The first part of Theorem 3.1 follows from the work of Stein and Street [22, 18] only when $p_0 = 1$. The more general statement gives a precise structural description of the L^2 boundedness properties for $H_{P,K,\mathcal{R}}$ and highlights the rôle of Heisenberg translations in multiparameter settings. Theorem 3.1 is a representative theorem and exposes a new phenomenon for multiparameter convolution operators on the Heisenberg group. More general results can be formulated and established.

[5]See Section 7 for a precise definition of the truncations.

4. Initial reductions for the proofs of Theorems 1.2 and 3.1.

We fix a product kernel K and use Proposition 2.1 to write $K = \sum_I \phi_I^{(I)}$ as in (4) with the smooth, compactly supported ϕ_I satisfying (5). We consider the operator

$$T_{P,\mathcal{F}} f(x,y,z) \;=\; \sum_{I\in\mathcal{F}} \int f((x,y,z)\cdot(s^{p_0},t,P(s,t))^{-1})\phi_I^{(I)}(s,t)\,ds\,dt$$

where $\mathcal{F} \subset \mathbb{Z}^2$ is a fixed finite subset $\mathcal{F} = \{I = (j,k)\}$, indexing the dyadic rectangles $\mathcal{R}_I = \{(s,t) : |s| \sim 2^j, |t| \sim 2^k\}$ in which $\phi_I^{(I)}$ (and hence the integral above) is supported. In Theorem 3.1, when P is assumed to be real-analytic near the origin, we require that the rectangles $\mathcal{R}_I$ be located near the origin; that is, if $I = (j,k) \in \mathcal{F}$, then both j and k are sufficiently negative.

In both Theorem 1.2 and Theorem 3.1, the operators we seek to bound are defined with respect to rough truncations over rectangles $\mathcal{R} = \{(s,t) : a \le |s| \le b, c \le |t| \le d\}$. In both cases, it suffices to consider the operator $T_{P,\mathcal{F}}$ defined with respect to smooth truncations and obtain bounds uniform in $\mathcal{F}$. In fact we can write

$$\iint_{\mathcal{R}} f((x,y,z)\cdot(s^{p_0},t,P(s,t))^{-1})\,K(s,t)\,dsdt \;=\; T_{P,\mathcal{F}}(x,y,z) + S$$

for some finite $\mathcal{F}$ and where S denotes a small sum of operators of the form

$$Tf(x,y,z) = \int_{a\le|s|\le 2a} \left[\int f((x,y,z)\cdot(s^{p_0},t,P(s,t))^{-1})\Big[\sum_{k\in\mathcal{F}_2}\psi_{j_1,k}(s,t)\Big]dt\right] ds, \tag{9}$$

together with one where the s integration is over $b/2 \le |s| \le b$ and others where the rôles of the s and t integrations are swapped. Here $\mathcal{F}_2$ is a finite subset of $\mathbb{Z}$ and $\psi_{j_1,k} = \phi_I^{(I)}$ where $I = (j_1,k)$. For fixed $a \le |s| \le 2a$, the kernel $K_{s,j_1}(t) = \sum_{k\in\mathcal{F}_2}\psi_{j_1,k}(s,t)$ is nonzero only if $2^{j_1} \sim |s|$ and defines a one-parameter Calderón-Zygmund kernel with constants controlled by $|s|^{-1}$; that is, for every $\ell \ge 0$, $|\partial_t^\ell K_{s,j_1}(t)| \lesssim |s|^{-1}|t|^{-\ell-1}$ and the crucial cancellation condition $\int K_{s,j_1}(t)dt = 0$ holds for every s and j_1.
Now fix x as well and write $g(y,z) = f(x - s^{p_0}, y, z)$ so that the integral in dt in (9) can be written as

$$Hg(y,z) \;=\; \int g\Big(y-t, z-Q(t)-\frac{1}{2}s^{p_0}y\Big)K_{s,j_1}(t)\,dt$$

where $Q(t) = P(s,t) + \frac{1}{2}xt$. But $\iint |Hg(y,z)|^2 dzdy = \iint |Lg(y,z)|^2 dzdy$ with

$$Lg(y,z) \;=\; \int g(y-t, z-Q(t))K_{s,j_1}(t)\,dt,$$

by a simple change of variables in the z integral. By the well-established one-parameter theory of singular Radon transforms (see for example [16]), we have uniform $L^2(\mathbb{R}^2)$ (in fact also L^p) bounds $\|Hg\|_2 \le C\|g\|_2$ where $C = C_s$ is independent of x, j_1 and is controlled by the Calderón-Zygmund constant of the kernel K_{s,j_1}; thus $C_s \lesssim |s|^{-1}$. Furthermore C can be taken to be independent of the coefficients of Q when P is a polynomial. By an application of Minkowski's integral inequality, uniform L^2 bounds for H imply uniform (uniform in the truncation $a \le |s| \le 2a$ and $\mathcal{F}_2$ as well) L^2 bounds for T in (9), so that the term S is taken care of.
Therefore it suffices to work with the operators $T_{P,\mathcal{F}}$ and obtain bounds which are uniform in $\mathcal{F}$.

By translation-invariance in the third variable we may assume, without loss of generality, that $P(0,0)=0$. Furthermore, the structure of the Heisenberg group allows us to make another reduction that will be very useful in the following. Rewriting P as $P(s,t)=cs^{p_0}+dt+\tilde{P}(s,t)$ with $\partial_s^{p_0}\tilde{P}(0,0)=\partial_t\tilde{P}(0,0)=0$, we see that we can also write

$$f((x,y,z)\cdot(s^{p_0},t,P(s,t))^{-1})=f(A[A^{-1}(x,y,z)\cdot(s^{p_0},t,\tilde{P}(s,t))^{-1}])$$

where

$$A \;=\; \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ c & d & 1\end{pmatrix}$$

is the *inner automorphism* of $\mathbb{H}^1$ determined by the element $(-d,c,0)$. Hence $\|T_{P,\mathcal{F}}\|_{L^2\to L^2}=\|T_{\tilde{P},\mathcal{F}}\|_{L^2\to L^2}$ and so we may assume in addition that

$$\partial_s^{p_0}P(0,0) \;=\; \partial_t P(0,0) \;=\; 0. \tag{10}$$

This innocent looking reduction will be fundamental later on, allowing us to estimate certain oscillatory integrals efficiently.

For Theorem 1.2, we take $p_0=1$, P a general real polynomial and $\mathcal{F}$ a general finite set of pairs (j,k) as specified above; our goal is to obtain $L^2(\mathbb{H}^1)$ bounds, uniform with respect to $\mathcal{F}$ and P lying in some subspace $\mathcal{V}_\Delta$ of real polynomials. For Theorem 3.1 we consider general $p_0\geq 1$ and real-analytic P near $(0,0)$, but we insist that the dyadic rectangles $\mathcal{R}_I$ associated to $I\in\mathcal{F}$ all lie in some small fixed neighbourhood (depending on P) of the origin $(0,0)$; no uniformity in P is sought in our L^2 bounds for the corresponding operators.

In analysing $T_{P,\mathcal{F}}$ we take an oscillatory integral approach. Viewing $T_{P,\mathcal{F}}f = f*_{\mathbb{H}^1}\mathcal{L}$ as a Heisenberg convolution operator, one can deduce via the group Fourier transform on $\mathbb{H}^1$, that

$$\|T_{P,\mathcal{F}}\|_{L^2(\mathbb{H}^1)\to L^2(\mathbb{H}^1)} \;\sim\; \sup_{\lambda\in\mathbb{R}}\|S^\lambda_{P,\mathcal{F}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}$$

where

$$S^\lambda_{P,\mathcal{F}}g(y) \;=\; \sum_{I\in\mathcal{F}}\int_{\mathbb{R}} m_I(\lambda,y,t)g(t)\,dt \;=:\; \sum_{I\in\mathcal{F}} S^\lambda_{P,I}g(y)$$

and

$$m_I(\lambda,y,t) \;=\; \int_{\mathbb{R}} e^{2\pi i\lambda(\frac{1}{2}(y+t)s^{p_0}+P(y-t,s))}\,\phi_I^{(I)}(s,y-t)\,ds.$$

See Ch. XII, §6.3 of [16] for an expression for the Fourier transform on $\mathbb{H}^1$.

Remark 2. Here we must caution the reader that the above reduction to a multiplier question on $\widehat{\mathbb{H}^1}$ does not come for free. Indeed, $\mathcal{L}$ above is a distribution and there is no a priori reason for it to have a well-behaved group Fourier transform. However, with a little care one can verify that the above reduction is indeed justified. For details, see for example [6] where an analogous one-parameter singular Radon transform is considered.

If $P(y-t,s)=\sum_{p,q\geq 0}c_{p,q}(y-t)^q s^p$, then since $P(0,0)=0$ and condition (10) holds, we can write $P(y-t,s)=\varphi(s)+\psi_0(y-t)+\sum_{p\geq 1}\psi_p(y-t)s^p$ where

$$\varphi(s)=\sum_{p\geq 1}c_{p,0}s^p,\quad \psi_0(y-t)=\sum_{q\geq 2}c_{0,q}(y-t)^q \quad\text{and}\quad \psi_p(y-t)=\sum_{q\geq 1}c_{p,q}(y-t)^q$$

so that $\psi_p(0)=0$ for all $p\geq 0$ (and $\psi_0'(0)=0$). Importantly we have $c_{p_0,0}=0$ (by (10)).

We can write the phase $\frac{1}{2}(y+t)s^{p_0}+P(y-t,s)$ of m_I as $ys^{p_0}+\tilde{P}(y-t,s)$ where the difference between $P(y-t,s)$ and $\tilde{P}(y-t,s)$ is that the coefficient $c_{p_0,1}$ in $\psi_{p_0}(y-t)$ is changed to $c_{p_0,1}-1/2$. This change does not affect P_{p_0} and so in the proofs of either Theorems 1.2 or 3.1 we may assume, without loss of generality, that

$$m_I(\lambda,y,t) \ = \ \int_{\mathbb{R}} e^{2\pi i\lambda(ys^{p_0}+P(y-t,s))}\,\phi_I^{(I)}(s,y-t)\,ds. \tag{11}$$

Clearly bounds on the oscillatory integral $\sum_{I\in\mathcal{F}} m_I(\lambda,y,t)$ will play a central rôle in our analysis. General estimates for oscillatory integrals will be detailed in the next section but for now we highlight a couple of generalisations (estimates (12) and (13) below which are proved in Section 5.1) of an important, well-known oscillatory integral bound due to Stein and Wainger [20] which states that for any real polynomial $Q\in\mathbb{R}[s]$, we have

$$\Bigl|\int_{a\le|s|\le b} e^{2\pi iQ(s)}\,\frac{ds}{s}\Bigr| \ \le \ C_d$$

where C_d depends only on the degree d of Q and is otherwise independent of the coefficients of Q as well as of a and b. A proof from a modern perspective is given in [16] and this perspective can be used to prove the stronger bound

$$\sum_{k\in S}|n_k(Q)| \ \le \ C_d \qquad \text{where} \qquad n_k(Q) \ = \ \int_{|s|\sim 2^k} e^{2\pi iQ(s)}\,\frac{ds}{s} \tag{12}$$

and S is any set of integers and C_d can be taken to be independent of S. A proof of (12) is given in Section 5.1 and is contained in the first part of the proof of the following bound.

In our context, we need to show that for any subset $\mathcal{F}'\subseteq\mathcal{F}$,

$$\sum_{I\in\mathcal{F}'}|m_I(\lambda,y,t)| \ \le \ C\,\frac{1}{|y-t|} \tag{13}$$

holds when either (i) P is a general real polynomial and $C=C_d$ depends only on the degree d of P (and in particular does not depend on the subset $\mathcal{F}'\subset\mathbb{Z}^2$, λ, y, t and the coefficients of P) or (ii) P is real-analytic near $(0,0)$ and $\mathcal{F}$ indexes dyadic rectangles $\mathcal{R}_I$ located near the origin; that is, the pairs $I=(j,k)$ range over integers $j\le -j_0$ and $k\le -k_0$ where j_0 and k_0 are large, fixed positive integers depending on our real-analytic function P. In this case, the constant C is allowed to depend on P and in particular it will depend on the truncation parameters j_0,k_0 but it does not depend on λ,y,t or the cardinality of $\mathcal{F}'$.

In Section 5.1 we will establish the estimate (13) in both cases but until then, we assume that it holds.

4.1. **Hilbert integral reduction.**

We assume that (13) holds. Choose $\chi\in C_0^\infty(\mathbb{R})$ supported in $\{|y|\sim 1\}$ and such that if $\chi_r(y):=\chi(2^{-r}y)$, we have $\sum_{r\in\mathbb{Z}}\chi_r(y)\equiv 1$ for $y\ne 0$. We decompose

$$S^\lambda_{P,\mathcal{F}}g(y) \ = \ \sum_{r\in\mathbb{Z}}\sum_{I\in\mathcal{F}}\chi_r(y)S^\lambda_{P,I}g(y) \ = \ S^1g(y)+S^2g(y)$$

where

$$S^1g(y) \ := \ \sum_{(r,I)\in\mathcal{G}^1}\chi_r(y)S^\lambda_{P,I}g(y) \qquad \text{and} \qquad S^2g(y) \ := \ \sum_{(r,I)\in\mathcal{G}^2}\chi_r(y)S^\lambda_{P,I}g(y)$$

and

$$\mathcal{G}^1 \ := \ \{(r,I) \in \mathbb{Z} \times \mathcal{F} : I = (j,k) \text{ satisfies } r \le k + C_0\}$$

for some large, fixed $C_0 > 0$. The set $\mathcal{G}^2$ is defined similarly but with the condition $k \le r - C_0$. The significance of this is that when $(r,I) \in \mathcal{G}^2$ we have $2^k \sim |y-t| \ll |y| \sim 2^r$.

Hence

$$|S^1 g(y)| \ \le \ \int \Big[\sum_{(r,I)\in\mathcal{G}^1} |\chi_r(y) m_I(\lambda, y, t)| \Big] \, |g(t)| \, dt$$

where the sum over $(r,I) \in \mathcal{G}^1$ is supported in $\{(y,t) : \delta|y| \le |y-t|\}$ for some small $\delta > 0$, depending on our choice of C_0. Using (13), we have

$$|S^1 g(y)| \ \le \ C \int_{\delta|y|\le|y-t|} \frac{1}{|y-t|} \, |g(t)| dt \ =: \ \int K(y,t) |g(t)| \, dt.$$

The integral operator with kernel K is of *Hilbert integral* type (the kernel is homogeneous of degree -1 and $K(1,t)|t|^{-1/2}$ is integrable over $\mathbb{R}$) and hence S^1 is uniformly bounded on $L^2(\mathbb{R})$ (uniform in λ, $\mathcal{F}$ and the coefficients of P in the polynomial case). See [15], page 271.

For S^2, write $S^2 g(y) = \sum_r S^2_r g(y)$ where

$$S^2_r g(y) \ = \ \sum_{I:(r,I)\in\mathcal{G}^2} \chi_r(y) S^\lambda_{P,I} g(y) \ =: \ \int K_r(y,t) g(t) \, dt.$$

For $|y| \sim 2^r$, we have $|t| = |t-y+y| \sim |y| \sim 2^r$ if $|y-t| \sim 2^k$ and $k \le r - C_0$. Hence $\mathrm{supp}(K_r) \subset \{(y,t) : |y|, |t| \sim 2^r\}$ and so

$$\|S^2\|_{L^2\to L^2} \ = \ \Big\| \sum_r S^2_r \Big\|_{L^2\to L^2} \ \sim \ \sup_r \|S^2_r\|_{L^2\to L^2} \tag{14}$$

by (almost) orthogonality. Therefore the proofs of both Theorem 1.2 and Theorem 3.1 reduce to understanding when the operators S^2_r are uniformly bounded on L^2.

5. Oscillatory integral estimates

Many oscillatory estimates rely on van der Corput's lemma which we now state.

van der Corput's Lemma. *For any $k \ge 2$, there exists a constant C_k such that*

$$\Big| \int_a^b e^{2\pi i \lambda \phi(s)} ds \Big| \ \le \ C_k |\lambda|^{-1/k}$$

holds for any real-valued $\phi \in C^k[a,b]$ such that $|\phi^{(k)}(s)| \ge 1$ for $s \in [a,b]$. The result holds for $k = 1$ if in addition we assume that ϕ' is monotone on $[a,b]$.

For a proof, see [16].

Let $Q(s) = \lambda[y s^{p_0} + P(y-t, s)]$ be the phase appearing in each

$$m_I(\lambda, y, t) \ = \ \int_{\mathbb{R}} e^{2\pi i Q(s)} 2^{-j-k} \phi_I(2^{-j}s, 2^{-k}(y-t)) \, ds \ =: \ 2^{-k} \mathcal{I}_j$$

where $I = (j,k)$ and

$$\mathcal{I}_j \ = \ \mathcal{I}_{j,\lambda,y,t,k} \ = \ \int_{\mathbb{R}} e^{2\pi i Q(s)} \, 2^{-j} \Phi(2^{-j}s) \, ds \tag{15}$$

is supported in $\{(y,t,k) \ : \ |y-t| \sim 2^k\}$. Here $\Phi(s) = \phi_I(s, 2^{-k}(y-t))$ is supported in $|s| \sim 1$ and has bounded C^ℓ norms, uniformly in the parameters y, t, I and k. Hence $|m_I(\lambda, y, t)| \le |\mathcal{I}_j| \ 2^{-k} \chi_{|y-t|\sim 2^k}$ and so to bound $\sum_I |m_I(\lambda, y, t)|$, it suffices to fix k and obtain uniform bounds for the sums $\sum_j |\mathcal{I}_j|$ over j. To do this, we will use van der Corput's lemma.

Our first application is a proof of (13).

5.1. **Proof of the generalised Stein-Wainger bound** (13)**.** Let $Q(s) = \lambda[ys^{p_0} + P(y-t, s)]$ be the phase appearing in each m_I and for each $k \in \mathbb{Z}$, set $\mathcal{F}'_k = \{j \in \mathbb{Z} : I = (j,k) \in \mathcal{F}'\}$. It suffices to show that for every $k \in \mathbb{Z}$,

$$\sum_{I \, : \, j \in \mathcal{F}'_k} |m_I(\lambda, y, t)| \ \lesssim \ 2^{-k} \chi_{|y-t|\sim 2^k} \tag{16}$$

since (13) follows by summing these estimates over $k \in \mathbb{Z}$. As observed above, this is equivalent to showing

$$\sum_{j \in \mathcal{F}'_k} |\mathcal{I}_j| \ \lesssim \ 1$$

where

$$\mathcal{I}_j \ := \ \int_{\mathbb{R}} e^{2\pi i Q(s)} \, 2^{-j} \Phi(2^{-j}s) \, ds \ = \ \int_{\mathbb{R}} e^{2\pi i Q_j(s)} \, \Phi(s) \, ds$$

and $Q_j(s) := Q(2^j s)$.

We start with the case when P is a polynomial where we seek bounds which are uniform in the coefficients of P, the subset $\mathcal{F}' \subseteq \mathcal{F}$, and the parameters λ, y and t. For the case when P is real-analytic at $(0,0)$, we will reduce the estimate (13) to the polynomial case.

In the polynomial case, our phase $Q(s) = \sum_{p\ge 1} e_p s^p$ is a polynomial (without loss of generality we may suppose that Q has no constant term) and hence $Q_j(s) = \sum_{p \ge 1} e_p 2^{pj} s^p$. A simple equivalence of norms argument shows that there exists a $c_d > 0$, depending only on the degree d of Q, such that for all j there exists ℓ_j with $1 \le \ell_j \le d$ for which $|Q_j^{(\ell_j)}(s)| \ge c_d \sum_{p\ge 1} |e_p| 2^{pj}$ holds on the support of Φ. An application of van der Corput's lemma now shows that $|\mathcal{I}_j| \le C_d (\Lambda_j)^{-1/\ell_j}$ where $\Lambda_j = \sum_{p \ge 1} |e_p| 2^{pj}$. Using (5),

$$\int_{\mathbb{R}} \Phi(s) \, ds \ = \ \int_{\mathbb{R}} \phi_I(s, 2^{-k}(y-t)) \, ds \ = \ 0,$$

one also has

$$|\mathcal{I}_j| = \Big| \int_{\mathbb{R}} [e^{2\pi i Q_j(s)} - 1] \Phi(s) ds \Big| \ \lesssim \ \Lambda_j \ \text{ and so } \ |\mathcal{I}_j| \lesssim \min(\Lambda_j, \Lambda_j^{-1/d}),$$

which allows us to sum in j to see $\sum_j |\mathcal{I}_j| \lesssim_d 1$, as desired.

Next we consider the real-analytic case so that the pairs $I = (j,k)$ in $\mathcal{F}$ range over integers $j \le -j_0$ and $k \le -k_0$ where j_0 and k_0 are large, fixed positive integers depending on our real-analytic function P. In this case, as said above, we will reduce matters to the polynomial case.

Recall our notation where we write $P(y-t,s) = \varphi(s) + \sum_{p\geq 0} \psi_p(y-t)s^p$ and $\varphi(s) = \sum_{p\geq 1} c_{p,0}s^p$. For $|y-t| \sim 2^k \ll 1$, we have $|\psi_p(y-t)| \lesssim_P 2^k$. Write

$$|m_I(\lambda,y,t)| = \Big|\int_{\mathbb{R}} e^{2\pi i\lambda[ys^{p_0}+\varphi(s)+\Psi_{y,t}(s)]}\, 2^{-j}\Phi(2^{-j}s)\, ds\Big|$$

where $\Psi_{y,t}(s) := \sum_{p\geq 1} \psi_p(y-t)s^p$. Hence $|\partial_s^p \Psi_{y,t}(s)| \lesssim_{p,P} 2^k$ for every $p \geq 0$ and in particular, $|\partial_s^p \Psi_{y,t}(s)| \ll_{p,P} 1$.

First we consider the case that there exists a $p_1 > p_0$ such that $c_{p_1,0} \neq 0$. Hence $|\varphi^{(p_1)}(s)| \gtrsim_P 1$ for $|s| \ll 1$ and so

$$|\partial_s^{p_1}[ys^{p_0} + \varphi(s) + \Psi_{y,t}(s)]| \;\gtrsim\; 1.$$

This puts us in a position to apply van der Corput's lemma, which together with a simple integration by parts argument allows us to conclude $|\mathcal{I}_j| \lesssim 2^{-j}|\lambda|^{-1/p_1}$ and so $\sum_{j\in S_\lambda} |\mathcal{I}_j| \lesssim 1$ where $S_\lambda = \{j : 2^j \geq |\lambda|^{-1/p_1}\}$. For $j \notin S_\lambda$, we compare the integral $\mathcal{I}_j$ to the integral

$$\mathcal{II}_j \;:=\; \int_{\mathbb{R}} e^{2\pi i\lambda[ys^{p_0}+\tilde\varphi(s)+\tilde\Psi_{y,t}(s)]}\, 2^{-j}\Phi(2^{-j}s)\, ds$$

where $\tilde\varphi(s) = \sum_{p=1}^{p_1-1} c_{p,0}s^p$ and $\tilde\Psi_{y,t}(s) = \sum_{p=1}^{p_1-1} \psi_p(y-t)s^p$. Note that the difference of the phases in $\mathcal{I}_j$ and $\mathcal{II}_j$ is at most $C|\lambda s^{p_1}|$ and so

$$|\mathcal{I}_j - \mathcal{II}_j| \;\lesssim\; |\lambda| 2^{p_1 j},$$

implying $\sum_{j\notin S_\lambda} |\mathcal{I}_j - \mathcal{II}_j| \lesssim 1$. We can appeal to our analysis of (13) when the phase is polynomial to conclude $\sum_{j\notin S_\lambda} |\mathcal{II}_j| \lesssim 1$ and hence (16) holds in this case.

Finally we consider the case $\varphi(s) = \sum_{p<p_0} c_{p,0}s^p$; that is, there is no $p_1 > p_0$ such that $c_{p_1,0} \neq 0$ (remember $c_{p_0,0} = 0$ by (10)). In this case we may suppose that there is a $p_1 > p_0$ such that $|\psi_{p_1}(y-t)| \sim |y-t|^{\ell_*}$ for some $\ell_* \geq 1$ and $\psi_p^{(\ell)}(0) = 0$ for all $p \geq p_1$ and all $\ell < \ell_*$. Indeed, either $\psi_p \equiv 0$ for all $p > p_0$ and we are back in the polynomial case, or not; in this second case, if we let $\ell_p := \min\{\ell : \psi_p^{(\ell)}(0) \neq 0\}$, we see that it suffices to take $\ell_* := \min_{p>p_0} \ell_p$ and p_1 a value that realises such minimum.
Thus in particular

$$\partial_s^{p_1}[ys^{p_0} + \varphi(s) + \Psi_{y,t}(s)] = c(y-t)^{\ell_*} + O((y-t)^{\ell_*}s)$$

and therefore $|\partial_s^{p_1}[ys^{p_0} + \varphi(s) + \Psi_{y,t}(s)]| \gtrsim 2^{\ell_* k}$ for $|s| \ll 1$ and $|y-t| \sim 2^k$. Hence by van der Corput's lemma, $|\mathcal{I}_j| \lesssim 2^{-j}[|\lambda|2^{\ell_* k}]^{-1/p_1}$ implying that $\sum_{j\in S'_\lambda} |\mathcal{I}_j| \lesssim 1$ where $S'_\lambda := \{j : 2^j \geq (|\lambda|2^{\ell_* k})^{-1/p_1}\}$. For $j \notin S'_\lambda$, we compare the integral $\mathcal{I}_j$ to the integral

$$\mathcal{III}_j \;:=\; \int_{\mathbb{R}} e^{2\pi i\lambda[ys^{p_0}+\varphi(s)+\tilde\Psi_{y,t}(s)]}\, 2^{-j}\Phi(2^{-j}s)\, ds$$

where again $\tilde\Psi_{y,t}(s) = \sum_{p=1}^{p_1-1} \psi_p(y-t)s^p$ as above. Note that the difference of the phases in $\mathcal{I}_j$ and $\mathcal{III}_j$ is at most $C|\lambda 2^{\ell_* k} s^{p_1}|$ and so

$$|\mathcal{I}_j - \mathcal{III}_j| \;\lesssim\; |\lambda| 2^{\ell_* k} 2^{p_1 j},$$

implying $\sum_{j\notin S'_\lambda} |\mathcal{I}_j - \mathcal{III}_j| \lesssim 1$. Once again we can appeal to our analysis of (13) when the phase is polynomial to conclude $\sum_{j\notin S'_\lambda} |\mathcal{III}_j| \lesssim 1$ and hence (16) holds in this case as well. This completes the proof of (13) in all cases.

5.2. **Another useful bound for oscillatory integrals.** A nontrivial application of van der Corput's lemma gives the following useful uniform bound for oscillatory integrals with polynomial phases.

Proposition 5.3. *For any $Q(s) = \sum_{j=1}^d h_j s^j \in \mathbb{R}[s]$ and $1 \le j \le d$, we have*

$$\Big|\int_{B/2}^{B} e^{2\pi i Q(s)}\, ds\Big| \ \le \ C_d |h_j|^{-1/d} B^{1-j/d}.$$

This is a simple variant of Theorem 3.1 in [8]. We have the following simple consequence for our multipliers $m_I(\lambda, y, t) = 2^{-k}\mathcal{I}_j$ when the phase $Q(s) = \lambda(ys^{p_0} + P(y-t, s)) = \sum_{p\le d} h_p s^p$ is a polynomial: for every $1 \le p \le d$

$$|\mathcal{I}_j| \lesssim_d \big[|h_p| 2^{pj}\big]^{-1/d}, \tag{17}$$

where we recall the definition of $\mathcal{I}_j$ in (15). We will use this estimate in the proof of Theorem 1.2 where P is a polynomial. For Theorem 3.1, when P is assumed to be real-analytic near $(0,0)$, we will need the following two variants of (17).

Consider again the phase

$$Q(s) = \lambda(ys^{p_0} + P(y-t, s)) = \lambda\Big[ys^{p_0} + \sum_{p\ge 1} c_{p,0} s^p + \sum_{p \ge 0} \psi_p(y-t) s^p\Big]$$

in $m_I(\lambda, y, t) = 2^{-k}\mathcal{I}_j$. The coefficient of s^{p_0} is $h_{p_0} = \lambda(y + \psi_{p_0}(y-t))$, again, since the coefficient $c_{p_0,0} = 0$ as per (10). This is important since it allows us to determine the size of h_{p_0}. In fact, for pairs $(r, I) \in \mathcal{G}^2$ arising in the definition of S_r^2, we have $|y - t| \sim 2^k \ll 2^r \sim |y|$ in the support of $\mathcal{I}_j = 2^{-k} m_I(\lambda, y, t)$ and so $|h_{p_0}| \sim |\lambda| 2^r$ since $\psi_{p_0}(y-t) = O_P(2^k)$. In this case, we have

$$|\mathcal{I}_j| \lesssim_P [|h_{p_0}| 2^{p_0 j}]^{-\epsilon} \sim [|\lambda| 2^r 2^{p_0 j}]^{-\epsilon} \tag{18}$$

for some $\epsilon > 0$.

Next we consider an estimate with respect to the coefficient $h_p = \lambda(c_{p,0} + \psi_p(y-t))$ of s^p in the phase $Q(s)$ for other values of p. In our arguments, this case will only arise in the simpler situation when the phase Q is truncated to either

$$\lambda\Big[(ys^{p_0} + \sum_{p \ge 1} c_{p,0} s^p + \sum_{1 \le p < p_0} \psi_p(y-t) s^p\Big]$$

or

$$\lambda\Big[(\sum_{p\ge 1} c_{p,0} s^p + \sum_{1 \le p < p_0} \psi_p(y-t) s^p\Big]$$

which is still not quite the case of a polynomial. For any $1 \le p < p_0$ with $\psi_p \not\equiv 0$, we have for some $\ell_p \ge 1$, $|\psi_p(y-t)| \sim 2^{\ell_p k}$ when $|y - t| \sim 2^k$. Hence the coefficient of $h_p = \lambda(c_{p,0} + \psi_p(y-t))$ satisfies $|h_p| \gtrsim 2^{\ell_p k}$ since $2^{\ell_p k} \ll 1$. In this situation, we have

$$|\mathcal{I}_j| \ \lesssim_{P,p} \ \big[|h_p| 2^{pj}\big]^{-\epsilon} \ \lesssim \ \big[|\lambda| 2^{\ell_p p} 2^{pj}\big]^{-\epsilon} \tag{19}$$

for some $\epsilon > 0$.

The proof of (18) is fairly simple and we present this case now. The proof of (19) is an elaboration on a proof of Proposition 5.3 and we have decided to give the proof in an appendix to the paper.

To prove (18) we begin as in the real-analytic case for (16) by initially assuming there exists a $p_1 > p_0$ such that $c_{p_1,0} \ne 0$. Hence $|\varphi^{(p_0)}(s)| \sim |s|^{p_1 - p_0}$ for $|s| \ll 1$

and so if $|s| \ll |y|^{1/(p_1-p_0)}$ or $|y|^{1/(p_1-p_0)} \ll |s|$ (that is, $2^j \not\sim 2^{r/(p_1-p_0)}$), we see that

$$|Q^{(p_0)}(s)| = |\lambda(p_0!\, y + \varphi^{(p_0)}(s) + O(2^k))| \ \gtrsim\ |\lambda|\big[|y| - C2^k\big] \ \gtrsim |\lambda||y| \ \sim\ |h_{p_0}|$$

since $2^k \ll 2^r \sim |y|$. Hence by van der Corput's lemma, we have $|\mathcal{I}_j| \lesssim_P 2^{-j}|h_{p_0}|^{-1/p_0}$ implying (18) with $\epsilon = 1/p_0$.

When $2^j \sim 2^{r/(p_1-p_0)}$, we consider the p_1-th derivative of Q: note that $|\varphi^{(p_1)}(s)| \sim_P 1$ for $|s| \ll 1$. Therefore we have

$$|Q^{(p_1)}(s)| = |\lambda(\varphi^{(p_1)}(s) + O(2^k))| \ \gtrsim\ |\lambda|$$

since $2^k \ll 1$. Hence van der Corput's lemma implies

$$|\mathcal{I}_j| \ \lesssim_P\ 2^{-j}|\lambda|^{-1/p_1} \ =\ (|\lambda|2^{j(p_1-p_0)}2^{p_0 j})^{-1/p_1} \ \sim\ (|h_{p_0}|2^{p_0 j})^{-1/p_1},$$

implying (18) with $\epsilon = 1/p_1$.

Finally we consider the case that for all $p_1 > p_0$ we have $c_{p_1,0} = 0$ in which case $\varphi^{(p_0)}(s) \equiv 0$ since we also have $c_{p_0} = 0$. Therefore as before,

$$|Q^{(p_0)}(s)| = |\lambda(p_0!\, y + O(2^k))| \ \gtrsim\ |\lambda|\big[|y| - C2^k\big] \ \gtrsim |\lambda||y| \ \sim\ |h_{p_0}|$$

since $2^k \ll 2^r \sim |y|$. Hence by van der Corput's lemma, we have $|\mathcal{I}_j| \lesssim_P (|h_{p_0}|2^{p_0 j})^{-1/p_0}$ implying (18) with $\epsilon = 1/p_0$. This completes the proof of (18) in all cases.

6. The proof of Theorems 1.2 and 3.1 – the main steps

In both Theorems 1.2 and 3.1, we need to establish uniform (in r) L^2 bounds for the operators

$$S_r^2 g(y) = \sum_{I:(r,I)\in\mathcal{G}^2} \chi_r(y) S_{P,I}^{\lambda} g(y) \quad \text{where} \quad S_{P,I}^{\lambda} g(y) = \int_{\mathbb{R}} m_I(\lambda, y, t) g(t)\, dt$$

and

$$m_I(\lambda, y, t) \ =\ \int_{\mathbb{R}} e^{2\pi i\lambda(ys^{p_0} + P(y-t,s))}\, \phi_I^{(I)}(s, y-t)\, ds.$$

See (14). Here

$$\mathcal{G}^2 \ =\ \{(r, I) \in \mathbb{Z}\times\mathcal{F} : I = (j,k) \text{ satisfies } k \le r - C_0\}$$

for some large, fixed $C_0 > 0$. Recall that we write

$$P(y-t, s) = \varphi(s) + \sum_{p\ge 0} \psi_p(y-t)s^p = \varphi(s) + \psi_0(y-t) + \sum_{p\in\mathcal{P}} \psi_p(y-t)s^p$$

where each $\psi_p(0) = 0$ and $\mathcal{P} := \{p \ge 1 : \psi_p \not\equiv 0\}$.

The plan of the proof is to use the oscillatory integral estimates discussed in Section 5 to bound the errors introduced when removing certain terms from the phase of m_I. We will keep removing terms from the phase whenever possible until we have reduced matters to (euclidean convolution) operators that are well-known already. These will be either (variable kernel) oscillatory singular integral operators à la Ricci-Stein [13] or the singular Radon transforms mentioned in the statements of Theorems 1.2 and 3.1.

6.1. **The exceptional set $\mathcal{E}$.** For both theorems, we will need to avoid an exceptional set $\mathcal{E}$ of bad values of k which we will make more and more explicit as we proceed. For Theorem 1.2, the cardinality $\#\mathcal{E} \lesssim_d 1$ will be bounded uniformly in $\mathcal{F}$ and the coefficients of P. For Theorem 3.1, the cardinality $\#\mathcal{E} \lesssim_P 1$ will depend on P (and hence on the truncation parameters j_0, k_0) but is otherwise independent of $\#\mathcal{F}$.

We split

$$S_r^2 g(y) \;=\; \sum_{I\in\mathcal{F}^{0,r}} \chi_r(y) S^{\lambda}_{P,I} g(y) + \sum_{I\in\mathcal{F}^{1,r}} \chi_r(y) S^{\lambda}_{P,I} g(y) \;:=\; S_r^{2,0} g(y) + S_r^{2,1} g(y)$$

where $\mathcal{F}^{0,r} = \{I = (j,k) : (r, I) \in \mathcal{G}^2, k \notin \mathcal{E}\}$ and $\mathcal{F}^{1,r}$ involves the bad values $k \in \mathcal{E}$. We use (16) with $\mathcal{F}' = \mathcal{F}^{1,r}$ to bound

$$|S_r^{2,1} g(y)| \le \sum_{k\in\mathcal{E}} \int \Big[\sum_{j\in\mathcal{F}^{1,r}_k} |m_I(\lambda, y, t)|\Big]\, |g(y)| dy \;\lesssim\; \sum_{k\in\mathcal{E}} 2^{-k} \int_{|y-t|\sim 2^k} |g(y)|\, dy$$

and so $\|S_r^{2,1}\|_{L^2\to L^2} \lesssim \#\mathcal{E} \lesssim 1$, leaving us with $S_r^{2,0}$ which avoids the bad values $k \in \mathcal{E}$.

To ease the notation, we rewrite $S_r^{2,0}$ as S_r^2 with the understanding that the sum defining S_r^2 is taken over $I = (j,k) \in \mathcal{F}^{0,r}$ and so $k \notin \mathcal{E}$.

For each term $\psi_{p_*}(y-t)$ with $p_* \in \mathcal{P}$ arising in the phase of m_I, our strategy is to reduce the analysis of S_r^2 to $S_r^{2,*} = \sum_{I\in\mathcal{F}^{0,r}} \chi_r(y) S^{\lambda,*}_{P,I}$ where

$$S^{\lambda,*}_{P,I} g(y) \;=\; \int_{\mathbb{R}} m_I^*(\lambda, y, t)\; g(t)\, dt$$

and

$$m_I^*(\lambda, y, t) \;=\; \int_{\mathbb{R}} e^{2\pi i \lambda (y s^{p_0} + \varphi(s) + \sum_{p\neq p_*} \psi_p(y-t) s^p)}\, \phi_I^{(I)}(s, y-t)\, ds; \tag{20}$$

that is, we plan to remove the term $\psi_{p_*}(y-t) s^{p_*}$ from the phase of m_I.

Our estimates are naturally expressed in terms of certain key quantities associated to the size of those $\psi_{p_*}(y-t)$ with $p_* \in \mathcal{P}$. For Theorem 3.1, when P is assumed to be real-analytic near $(0,0)$, we can find an $\ell_* \ge 1$ such that $|\psi_{p_*}(y-t)| \sim c_* 2^{\ell_* k}$ when $|y-t| \sim 2^k \ll 1$. This simply follows from the fact that $\psi_{p_*}(0) = 0$ and $\psi_{p_*} \not\equiv 0$. For Theorem 1.2 the $\psi_p(y-t)$ are general polynomials and $|y-t| \sim 2^k$ can be of any size ($k \in \mathbb{Z}$ can take any value). Here we will appeal to a result in [1] which shows that outwith finitely many values of k (depending only on the degree of P), there exists an $\ell_* \ge 1$ such that indeed $|\psi_{p_*}(y-t)| \sim c_* 2^{\ell_* k}$ when $|y-t| \sim 2^k$.

Given a nonzero polynomial $Q(t) \in \mathbb{R}[t]$, a basic result in [1] gives us a decomposition $\mathbb{R} = S \cup G$ where $S = \cup J$ can be written as a disjoint union of $O(1)$ (with constant only depending on the degree of Q) intervals such that on each J, $|Q(t)| \sim c_J |t|^{\ell_J}$ for some $\ell_J \in \mathbb{N}$. Furthermore if $Q(0) = 0$, then $\ell_J \ge 1$ for all J. Finally each interval comprising $G = \mathbb{R} \setminus S$ is a dyadic interval of the form $[A, CA]$ where $C \lesssim 1$.

As above, we write our polynomial P as $P(s,t) = \varphi(s) + \sum_{p\ge 0} \psi_p(t) s^p$ where each $\psi_p \in \mathbb{R}[t]$ satisfies $\psi_p(0) = 0$. We apply the decomposition in [1] to each ψ_p with $p \in \mathcal{P}$ (so that $\psi_p \not\equiv 0$) to conclude that there is an exceptional set $\mathcal{B}$ of $O_d(1)$

values of k where $\mathbb{Z} \setminus \mathcal{B} = \cup_{n=1}^N S_n$ decomposes into $O_d(1)$ sets such that for each $p \in \mathcal{P}$ and n, there is an $\ell_p = \ell_{p,n} \geq 1$ and $c_p = c_{p,n} > 0$ with the property that

$$|\psi_p(y-t)| \sim c_p \, 2^{\ell_p k} \quad \text{whenever} \quad |y-t| \sim 2^k \quad \text{and} \quad k \in S_n. \tag{21}$$

We incorporate the set $\mathcal{B}$ into $\mathcal{E}$ so that $I = (j,k) \in \mathcal{F}^{0,r}$ implies $k \in S_n$ for some n and (21) holds for every ψ_p with $p \in \mathcal{P}$.

6.2. **Key quantities and the first step.** The key quantities $\mathcal{A}_{p_*}(k) = \mathcal{A}_{p_*,\lambda,r}(k)$ are defined as

$$\mathcal{A}_{p_*}(k) \; := \; \frac{|\lambda| c_* 2^{\ell_* k}}{(|\lambda| 2^r)^{p_*/p_0}}$$

where, in the case of Theorem 1.2, $c_* = c_{p_*}$ and $\ell_* = \ell_{p_*}$ appear in (21). One important estimate where these quantities arise occurs in the following bound for the differences $D_k := \sum_{j \in \mathcal{F}_k^{0,r}} [m_I - m_I^*]$ (which avoids the exceptional values of $k \in \mathcal{E}$),

$$|D_k| \; \lesssim \; \mathcal{A}_{p_*}(k)^{\epsilon_*} \;\; 2^{-k} \chi_{|y-t| \sim 2^k} \tag{22}$$

for some $\epsilon_* > 0$. We prove this bound below.

For Theorem 1.2, the implicit constant in the estimate (22) will be uniform; it will depend only on the degree of P and can be taken to be independent of the coefficients of P as well as the set $\mathcal{F}$. For Theorem 3.1 the implicit constant will depend on P.

To prove (22), we split (recall the definition of m_I^* in (20))

$$D_k \; = \; \sum_{j \in J_1} [m_I - m_I^*] \; + \; \sum_{j \in J_2} [m_I - m_I^*] \; =: \; D_k^1 + D_k^2$$

where $J_1 \sqcup J_2 = \{ j : I = (j,k) \in \mathcal{F}^{0,r} \}$ and

$$J_1 \; := \; \{ j : I = (j,k) \in \mathcal{F}^{0,r} \text{ and } 2^j \leq (|\lambda| 2^r)^{-1/p_0} \, \mathcal{A}_{p_*}(k)^{-\sigma} \}$$

for some $\sigma > 0$ to be chosen later. For $j \in J_1$, we use that the difference in the phases of m_I and m_I^* is at most $C |\lambda| c_* 2^{\ell_* k} 2^{p_* j}$ (the constant C being absolute/uniform) to conclude that $|D_k^1| \leq$

$$\sum_{j \in J_1} |m_I - m_I^*| \; \lesssim \; 2^{-k} \, \chi_{|y-t| \sim 2^k} \; |\lambda| c_* 2^{\ell_* k} \sum_{j \in J_1} 2^{p_* j} \; \lesssim \; \mathcal{A}_{p_*}(k)^{\epsilon_*} \;\; 2^{-k} \chi_{|y-t| \sim 2^k}$$

where $\epsilon_* = 1 - \sigma p_* > 0$ and we have chosen $\sigma < 1/p_*$; this shows that (22) holds for D_k^1.

For D_k^2, we treat m_I and m_I^* separately, bounding $|D_k^2| \leq \sum_{j \in J_2} |m_I| + \sum_{j \in J_2} |m_I^*|$. Recall that

$$|m_I| \; = \; 2^{-k} \Big| \int_{\mathbb{R}} e^{2\pi i \lambda (y s^{p_0} + \varphi(s) + \sum_{p \geq 1} \psi_p(y-t) s^p)} 2^{-j} \Phi(2^{-j} s) \, ds \Big|.$$

We will apply (17) and (18) to

$$Q(s) = \lambda \Big[(y + \psi_{p_0}(y-t)) s^{p_0} + \varphi(s) + \sum_{p \neq p_0} \psi_p(y-t) s^p \Big]$$

with respect to the coefficient $h_{p_0} := \lambda(y + \psi_{p_0}(y-t))$ of s^{p_0}. Very importantly, we have reduced (see (10)) to the case where the coefficient $c_{p_0,0}$ in $\varphi(s) = \sum_{p \geq 1} c_{p,0} s^p$ is zero!

If $\psi_{p_0}(y-t) \equiv 0$, then $h_{p_0} = \lambda y$ and if $\psi_{p_0}(y-t) \not\equiv 0$, then for $k \notin \mathcal{E}$, we have $|\psi_{p_0}(y-t)| \sim c_0 2^{\ell_0 k}$ for some $\ell_0 \geq 1$ when $|y-t| \sim 2^k$. Hence there are only $O(1)$ values of k where the bound $|h_{p_0}| \sim |\lambda y| \sim |\lambda| 2^r$ does not hold. We add these values to the expectional set $\mathcal{E}$. Hence (17) and (18) imply

$$|m_I| \lesssim \big[|\lambda| 2^r 2^{p_0 j}\big]^{-\epsilon_0} \, 2^{-k} \chi_{|y-t| \sim 2^k}$$

for some $\epsilon_0 > 0$. The same argument shows that $|m_I^*|$ satisfies this estimate as well. Summing over $j \in J_2$ establishes (22) for D_k^2 and hence D_k.

6.3. **An interlude – some analysis specific to Theorem 3.1.** For Theorem 3.1 (in which case both $j, k \leq 0$ for $I = (j,k) \in \mathcal{F}$), we claim that when $p_* > p_0$, the above differences D_k also satisfy

$$|D_k| \lesssim_P \mathcal{A}_{p_*}(k)^{-\epsilon_*} \, 2^{-k} \chi_{|y-t| \sim 2^k} \tag{23}$$

where $\epsilon_* = p_0/(p_* - p_0) > 0$. This, together with (22), will allow us to remove all terms $\psi_p(y-t) s^p$ with $p \geq p_0$ from the phase of m_I.

The proof of (23) is straightforward. We again use that the difference in the phases of m_I and m_I^* is at most $C|\lambda| 2^{\ell_* k} 2^{p_* j}$ (the constant C being absolute/uniform) to conclude that $|D_k| \leq$

$$\sum_{j : I = (j,k) \in \mathcal{F}^{0,r}} |m_I - m_I^*| \lesssim_P |\lambda| 2^{\ell_* k} \Big[\sum_{j \leq 0} 2^{p_* j}\Big] \, 2^{-k} \chi_{|y-t| \sim 2^k} \lesssim |\lambda| 2^{\ell_* k} \, 2^{-k} \chi_{|y-t| \sim 2^k}.$$

However for $I = (j,k) \in \mathcal{F}^{0,r}$ we have $k \leq 0$ and $k \leq r$ and hence it can be verified that

$$|\lambda| 2^{\ell_* k} \leq |\lambda| 2^k \leq \Big[\frac{(|\lambda| 2^r)^{p_*/p_0}}{|\lambda| 2^{\ell_* k}}\Big]^{p_0/(p_* - p_0)}.$$

Therefore $|\lambda| 2^{\ell_* k} \leq \mathcal{A}_{p_*}(k)^{-\epsilon_*}$ and so (23) follows.

Note that when $k \leq 0$ and $I = (j,k) \in \mathcal{F}^{0,r}$ (and so $k \ll r$), we have

$$\mathcal{A}_{p_0}(k) = c_0 \, 2^{\ell_0 k} 2^{-r} \leq c_0 \, 2^k 2^{-r} \ll 1.$$

Putting (22) and (23) together, we see that in the situation of Theorem 3.1 and when $p_* \geq p_0$, the differences satisfy

$$|D_k| \lesssim_P \min(\mathcal{A}_{p_*}(k), \mathcal{A}_{p_*}(k)^{-1})^{\epsilon_*} \, 2^{-k} \chi_{|y-t| \sim 2^k}$$

for some $\epsilon_* > 0$. This allows us to sum over k and conclude that

$$\|S_r^2 - S_r^{2,*}\|_{L^2 \to L^2} \lesssim_P \sum_k \min(\mathcal{A}_{p_*}(k), \mathcal{A}_{p_*}(k)^{-1})^{\epsilon_*} \lesssim_P 1,$$

reducing matters to bounding $S_r^{2,*}$, uniformly in r - in other words, we have safely removed term $\psi_{p_*}(y-t) s^{p_*}$ from the phase.

We can now apply this argument iteratively, comparing $S_r^{2,*}$ to $S_r^{2,**}$ where the phase in $S_r^{2,**}$ has both ψ_{p_*} and $\psi_{p_{**}}$ removed and $p_*, p_{**} \geq p_0$. Notice though that the same argument above also allows us to remove an entire tail

$$\tilde{\psi}_{p_1}(y-t, s) = \sum_{p \geq p_1} \psi_p(y-t) s^p \quad \text{for some} \quad p_1 \geq p_0.$$

In fact we may suppose that there is a $p_1 \geq p_0$ such that $|\psi_{p_1}(y-t)| \sim_P |y-t|^{\ell_1}$ for some $\ell_1 \geq 1$ and $\psi_p^{(\ell)}(0) = 0$ for all $p \geq p_1$ and all $\ell < \ell_1$. Otherwise $\psi_p \equiv 0$ for all $p \geq p_0$ and so $\tilde{\psi}_{p_0} \equiv 0$. Hence $|\tilde{\psi}_{p_1}(y-t, s)| \sim c_1 |(y-t)^{\ell_1} s^{p_1}|$ for some c_1 and so $\tilde{\psi}_{p_1}(y-t, s)$ can be treated in the same way as $\psi_{p_1}(y-t) s^{p_1}$ and thus be removed from the phase. The above iteration then removes the remaining terms with $p_0 \le p_* < p_1$.

Hence for Theorem 3.1, the uniform (in r) L^2 boundedness of S_r^2 is equivalent to the uniform (in r) L^2 boundedness of

$$H_r g(y) \ := \ \chi_r(y) \sum_{I \in \mathcal{F}^{0,r}} \int_{\mathbb{R}} \rho_I(\lambda, y, t)\, g(t)\, dt$$

where

$$\rho_I(\lambda, y, t) \ = \ \int_{\mathbb{R}} e^{2\pi i \lambda (y s^{p_0} + \varphi(s) + \sum_{0 \le p < p_0} \psi_p(y-t) s^p)}\, \phi_I^{(I)}(s, y-t)\, ds.$$

Note that $P_{p_0}(s,t) = \varphi(s) + \sum_{0 \le p < p_0} \psi_p(t) s^p$ is precisely the function featuring in the statement of Theorem 3.1.

6.4. **Back to the common analysis of Theorems 1.2 and 3.1.** To unify the notation somewhat, we will designate as $\mathcal{H}_r$ both the operator

$$S_r^2 g(y) \ = \ \chi_r(y) \sum_{I \in \mathcal{F}^{0,r}} \int_{\mathbb{R}} m_I(\lambda, y, t)\, g(t)\, dt$$

when we refer to Theorem 1.2 and the operator H_r in the previous section defined with ρ_I instead of m_I when we refer to Theorem 3.1. Furthermore we relabel ρ_I as m_I so that when we refer to Theorem 3.1,

$$m_I(\lambda, y, t) \ = \ \int_{\mathbb{R}} e^{2\pi i \lambda (y s^{p_0} + \varphi(s) + \sum_{0 \le p < p_0} \psi_p(y-t) s^p)}\, \phi_I^{(I)}(s, y-t)\, ds$$

and when we refer to Theorem 1.2,

$$m_I(\lambda, y, t) \ = \ \int_{\mathbb{R}} e^{2\pi i \lambda (y s + \varphi(s) + \sum_{p \ge 0} \psi_p(y-t) s^p)}\, \phi_I^{(I)}(s, y-t)\, ds.$$

Of course the functions in the phase of m_I are real-analytic for Theorem 3.1 and they are polynomials for Theorem 1.2.

We split the operator $\mathcal{H}_r = \mathcal{H}_r^1 + \mathcal{H}_r^2$ where

$$\mathcal{H}_r^1 g(y) \ := \ \chi_r(y) \sum_{I \in \mathcal{F}_1^{0,r}} \int_{\mathbb{R}} m_I(\lambda, y, t)\, g(t)\, dt,$$

with

$$\mathcal{F}_1^{0,r} = \{I = (j,k) \in \mathcal{F}^{0,r} : k \in K_1\} \ \text{ and } \ K_1 = \{k : \mathcal{A}_p(k) \le 1, \text{for all } p \in \mathcal{P}\}.$$

The operator $\mathcal{H}_r^2$ is defined similarly where the k sum with $I \in \mathcal{F}_2^{0,r}$ is taken over the complementary set K_2 where at least one $p \in \mathcal{P}$ satisfies $\mathcal{A}_p(k) \ge 1$.

For $\mathcal{H}_r^1$, we proceed as in Section 6.2, using (22) to bound the difference $\mathcal{H}_r^1 - \mathcal{H}_r^{1,*}$ where $\mathcal{H}_r^{1,*}$ is defined the same as $\mathcal{H}_r^1$ except with m_I replaced by m_I^* – see (20) (of course for Theorem 3.1, we need to adjust appropriately the phase in m_I^* – we also note that the difference bound (22) still holds for $m_I - m_I^*$ in the context of Theorem 3.1). Hence (22) implies that

$$\|\mathcal{H}_r^1 - \mathcal{H}_r^{1,*}\|_{L^2 \to L^2} \ \lesssim \ \sum_{k \in K_1} \mathcal{A}_{p_*}(k)^{\epsilon_*} \ \lesssim \ 1.$$

Proceeding iteratively, we see that the uniform boundedness of $\mathcal{H}_r^1$ is reduced to the uniform boundedness of

$$L_r g(y) \;:=\; \chi_r(y) \sum_{I\in\mathcal{F}_1^{0,r}} \int_{\mathbb{R}} \tau_I(\lambda,y,t)\, e^{2\pi i\lambda\psi_0(y-t)} g(t)dt$$

where

$$\tau_I(\lambda,y,t) \;=\; \int_{\mathbb{R}} e^{2\pi i\lambda(ys^{p_0}+\varphi(s))}\, \phi_I^{(I)}(s,y-t)\,ds.$$

We note that $L_r g(y) = \chi_r(y)\int K_r(y,y-t)e^{2\pi i\lambda\psi_0(y-t)}g(t)\,dt$ where

$$K_r(y,y-t) \;=\; \sum_{k\in K_1} 2^{-k}K_r^{(k)}(y,2^{-k}(y-t))$$

and

$$K_r^{(k)}(y,\tau) \;=\; \sum_{j:I=(j,k)\in\mathcal{F}_1^{0,r}} \int_{\mathbb{R}} e^{2\pi i\lambda(ys^{p_0}+\varphi(s))}2^{-j}\phi_I(2^{-j}s,\tau)\,ds$$

Hence K_r is a variable Calderón-Zygmund kernel on $\mathbb{R}$; that is,

$$\int_{\mathbb{R}} K_r(y,\tau)\,d\tau \;=\; 0 \quad \text{ for all } y, \text{ and } \quad |\partial_\tau^\ell K_r(y,\tau)| \lesssim |\tau|^{-\ell-1} \quad \text{ for all } \ell, \tag{24}$$

uniformly in r and y. This follows from an simple variant of (13); more precisely, one sees that (16) remains true with ϕ_I replaced by any derivative $\partial_t^k\phi_I(s,t)$.

This puts us in a position to appeal to a theorem of Ricci and Stein in [13] on uniform L^2 bounds for oscillatory singular integral operators

$$T_\lambda g(y) \;=\; \int_{\mathbb{R}} K(y-t)e^{i\lambda\psi_o(y-t)}\,g(t)\,dt.$$

When ψ_0 is a polynomial (which is the case for Theorem 1.2), Ricci and Stein establish L^2 bounds which are uniform in λ, the Calderón-Zgymund kernel K and the coefficients of ψ_0. In [9], Pan extended this result to real-analytic phases ψ_0 (the case for Theorem 3.1). Although their results are stated and proved for classical Calderón-Zygmund kernels, an examination of their arguments shows that the same results hold for variable Calderón-Zygmund kernels described above in (24). At the heart of their argument is a $T_\lambda^* T_\lambda$ argument applied to dyadic pieces of the operator. Fortunately the order of the composition is immaterial (in fact they chose the order $T_\lambda^* T_\lambda$) but for our variable Calderón-Zygmund kernel K_r above, it is important to take the order $T_\lambda T_\lambda^*$ so that the variable y in the first argument of $K_r(y,y-t)$ does not interact with the integration defining the kernels of the various $T_\lambda T_\lambda^*$s. We leave the details to the reader. This completes the analysis for the $\mathcal{H}_r^1$; they define uniformly bounded L^2 operators.

For $\mathcal{H}_r^2$, our goal will be to establish uniform L^2 bounds for the difference $\mathcal{H}_r^2 - T_r'$ where T_r' is defined exactly the same as $\mathcal{H}_r^2$ except that $m_I(\lambda,y,t)$ is replaced by

$$e_I(\lambda,y-t) \;=\; \int_{\mathbb{R}} e^{2\pi i\lambda(\varphi(s)+\sum_{p\ge 0}\psi_p(y-t)s^p)}\,\phi_I^{(I)}(s,y-t)\,ds$$

for Theorem 1.2 and

$$e_I(\lambda,y-t) \;=\; \int_{\mathbb{R}} e^{2\pi i\lambda(\varphi(s)+\sum_{0\le p<p_0}\psi_p(y-t)s^p)}\,\phi_I^{(I)}(s,y-t)\,ds$$

for Theorem 3.1. That is, for $\mathcal{H}_r^2$ we plan to remove the term ys^{p_0} from the phase this time. Note that the phase in the first integral is precisely the original $P(s,y-t)$.

It is a simple matter to see that uniform boundedness of the family $\{T'_r\}$ is equivalent to the uniform boundedness of the euclidean translation-invariant family $\{T_r\}$ where

$$T_r g(y) \;=\; K_r * g(y) \;\text{ and }\; K_r(\tau) \;=\; \sum_{I \in \mathcal{F}_2^{0,r}} e_I(\lambda, \tau);$$

thus T_r is the same at T'_m without the $\chi_r(y)$ factor in front.

In fact from the pointwise bound $|T'_r g(y)| \le |T_r g(y)|$, one direction is clear. Suppose now that the family $\{T'_r\}$ is uniformly bounded in L^2 and decompose an $L^2(\mathbb{R})$ function $g = \sum_\ell g_\ell$ so that the support of $\tilde{g}_\ell(t) := g_\ell(\ell 2^r + t)$ is contained in $\{|t| \sim 2^r\}$. Since for $I = (j,k) \in \mathcal{F}_2^{0,r}$, $k \ll r$, we see that if $|y - t| \sim 2^k$ and $|t| \sim 2^r$, then $|y| \sim 2^r$ and so

$$T_r g_\ell(y + \ell 2^r) \;=\; T_r \tilde{g}_\ell(y) \;=\; \chi_r(y) T_r \tilde{g}_\ell(y) \;=\; T'_r \tilde{g}_\ell(y).$$

Therefore, by almost disjointness of the supports,

$$\|T_r g\|_{L^2}^2 \lesssim \sum_\ell \|T_r g_\ell\|_{L^2}^2 = \sum_\ell \|T'_r \tilde{g}_\ell\|_{L^2}^2 \lesssim \sum_\ell \|\tilde{g}_\ell\|_{L^2}^2 = \sum_\ell \|g_\ell\|_{L^2}^2 = \|g\|_{L^2}^2.$$

The difference $\mathcal{H}_r^2 - T'_r$ is

$$(\mathcal{H}_r^2 - T'_r) g(y) = \chi_r(y) \int_{\mathbb{R}} \sum_{I \in \mathcal{F}_2^{0,r}} \left[m_I(\lambda, y, t) - e_I(\lambda, y - t) \right] g(t)\, dt$$

and so we concentrate on bounding the difference

$$\mathcal{D} \;:=\; \sum_{I \in \mathcal{F}_2^{0,r}} [m_I - e_I] \;=\; \sum_{k \in K_2} \sum_{j : I = (j,k) \in \mathcal{F}_2^{0,r}} [m_I - e_I] \;=:\; \sum_{k \in K_2} \mathcal{D}_k.$$

We split $K_2 = \bigcup_{p \in \mathcal{P}} K_{2,p}$ where

$$K_{2,p} \;:=\; \left\{ k \in K_2 : \mathcal{A}_p(k) \ge \mathcal{A}_{p'}(k), \ \forall\, p' \in \mathcal{P} \right\}$$

so that when $k \in K_{2,p}$, we have $\mathcal{A}_p(k) \ge 1$ (by definition of K_2). This gives a corresponding splitting of $\mathcal{H}_r^2 - T'_r = \sum_{p \in \mathcal{P}} (\mathcal{H}_r^2 - T'_r)_p$ where the summation over $I = (j,k) \in \mathcal{F}_2^{0,r}$ is restricted to $k \in K_{2,p}$.

We claim that for $k \in K_{2,p}$,

$$|\mathcal{D}_k| \;\lesssim_P\; \mathcal{A}_p(k)^{-\epsilon_p}\, 2^{-k}\, \chi_{|y-t| \sim 2^k} \tag{25}$$

for some $\epsilon_p > 0$. If this is the case, then we have

$$\|(\mathcal{H}_r^2 - T'_r)_p\|_{L^2 \to L^2} \;\lesssim_P\; \sum_{k \in K_{2,p}} \mathcal{A}_p(k)^{-\epsilon_p} \;\lesssim\; 1$$

and so summing over $p \in \mathcal{P}$ gives the desired uniform bound for $\mathcal{H}_r^2 - T'_r$.

To prove (25), we fix p and $k \in K_{2,p}$ and split

$$\mathcal{D}_k \;= \sum_{j \in J_1} [m_I - e_I] \;+\; \sum_{j \in J_2} [m_I - e_I] \;:=\; \mathcal{D}_k^1 + \mathcal{D}_k^2$$

into two parts; here $J_1 = \{ j : 2^j \le (|\lambda| 2^r)^{-1/p_0} \mathcal{A}_p(k)^{-\sigma_p} \}$ for some $\sigma_p > 0$ and J_2 is the complementary range.

For $\mathcal{D}_k^1$, we use the difference in the phases of m_I and e_I to see that

$$|m_I(\lambda, y, t) - e_I(\lambda, y - t)| \;\lesssim\; |\lambda y| 2^{j p_0} \;\sim\; |\lambda| 2^r 2^{j p_0}$$

and so

$$|\mathcal{D}_k^1| \lesssim |\lambda| 2^r \Big[\sum_{j\in J_1} 2^{jp_0}\Big] 2^{-k}\chi_{|y-t|\sim 2^k} \lesssim \mathcal{A}_p(k)^{-p_0\sigma_p}\ 2^{-k}\chi_{|y-t|\sim 2^k},$$

establishing (25) for $\mathcal{D}_k^1$. For $\mathcal{D}_k^2$ we treat the terms m_I and e_I separately, bounding $|\mathcal{D}_k^2| \le \sum_{j\in J_2}|m_I| + \sum_{j\in J_2}|e_I|$.

We will apply both (17) and (19) to each m_I and e_I separately. The phase in m_I is

$$\lambda\Big[ys^{p_0} + \sum_{p\ge 1} c_{p,0}s^p + \sum_{p\ge 0}\psi_p(y-t)s^p\Big]$$

for Theorem 1.2, whereas for Theorem 3.1 the sum $\sum_{p=0}^{p_0-1}\psi_p(y-t)s^p$ is truncated. The phase in e_I is the same except the term ys^{p_0} is not present. They both have the s^p coefficient $h_p := \lambda(c_{p,0} + \psi_p(y-t))$ unless $p=1$ and we are in the setting of Theorem 1.2. Setting this case aside for the moment, we apply (17) and (19) to each m_I and e_I with respect to this common coefficient h_p. Since for some $\ell_p \ge 1$, $|\psi_p(y-t)| \sim c_p 2^{\ell_p k}$ when $|y-t|\sim 2^k$, we see that there are only $O(1)$ values of k where the bound $|h_p| \sim |\lambda| c_p 2^{\ell_p k}$ does not hold. We add these values to the expectional set $\mathcal{E}$. Hence in this case, (17) and (19) imply

$$|m_I|, |e_I| \lesssim \big[|\lambda| c_p 2^{\ell_p k} 2^{pj}\big]^{-\epsilon_0}\ 2^{-k}\chi_{|y-t|\sim 2^k} \tag{26}$$

for some $\epsilon_0 > 0$.

If in the context of Theorem 1.2 (so that $p_0 = 1$ and hence the coefficient $c_{1,0}$ in $\varphi(s)$ is zero) we are considering the case $p=1$, observe that the coefficient of s for m_I, which is $h_1 = \lambda(y + \psi_1(y-t))$, is different from the coefficient of s for e_I, $h_1 = \lambda\psi_1(y-t)$. However in both cases, except for a few values of k (which we toss into $\mathcal{E}$), we have $|h_1| \gtrsim |\lambda| c_1 2^{\ell_1 k}$ and so the estimate (26) holds in this case as well if one chooses σ_1 so that $0 < \sigma_1 < 1$.

Summing the estimates (26) over $j \in J_2$ establishes (25) for $\mathcal{D}_k^2$ and hence $\mathcal{D}_k$. This shows that the uniform L^2 boundedness of $\mathcal{H}_r$ is equivalent to the uniform L^2 boundedness of T_r.

Putting everything together, we see that the L^2 boundedness of the original convolution operator $T_{P,\mathcal{F}}$ on the Heisenberg group $\mathbb{H}^1$ is equivalent to the uniform in r (and λ) L^2 boundedness of the euclidean convolution operators T_r. Recall the definition of the operators T_r differs depending on whether we are in the context of Theorem 1.2 or Theorem 3.1. In the context of Theorem 1.2, the multiplier for T_r is

$$\int_{\mathbb{R}} K_r(t)e^{2\pi i\eta t}\,dt = \sum_{I\in\mathcal{F}_2^{0,r}} \iint_{\mathbb{R}^2} e^{2\pi i(\eta t + \lambda P(s,t))}\phi_I^{(I)}(s,t)\,dsdt$$

and so the uniform L^2 boundedness of the T_r is equivalent to showing that the above sum of integrals is bounded uniformly in the parameters r, λ and η.
In the context of Theorem 3.1, the multiplier for T_r is

$$\widehat{K_r^\lambda}(\eta) = \int_{\mathbb{R}} K_r(t)e^{2\pi i\eta t}\,dt = \sum_{I\in\mathcal{F}_2^{0,r}} \iint_{\mathbb{R}^2} e^{2\pi i(\eta t + \lambda P_{p_0}(s,t))}\phi_I^{(I)}(s,t)\,dsdt$$

and uniform boundedness is equivalent to showing that $\widehat{K_r^\lambda}(\eta)$ is uniformly bounded in r, λ and η.

7. The conclusion of the proof of Theorem 3.1

Consider the following truncations of the multiparameter singular Radon transform $R_{P_{p_0},K}$ (from the statement of Theorem 3.1):

$$R_{P_{p_0},K_r} f(x,y) \; = \; \iint_{\mathbb{R}^2} f(x-t, y-P_{p_0}(s,t))\, K_r(s,t) ds dt$$

where

$$K_r(s,t) \; = \; \sum_{I \in \mathcal{F}_2^{0,r}} \phi_I^{(I)}(s,t)$$

is a truncation of the product kernel K. The multiplier $M_r(\eta,\lambda)$ of $R_{P_{p_0},K_r}$ is precisely equal to $\widehat{K_r^\lambda}(\eta)$ above.

Thus the $L^2(\mathbb{H}^1)$ boundedness of $T_{P,\mathcal{F}}$ is equivalent to the uniform $L^2(\mathbb{R}^2)$ boundedness of the truncations $R_{P_{p_0},K_r}$ as stated in Theorem 3.1. When $K(s,t) = \mathcal{K}(s,t) = 1/st$ is the double Hilbert transform kernel, the operator $R_{P_{p_0},K}$ and its generalisations have been thoroughly investigated in several papers; see for example, [3], [2], [10] and [11]. In [3] it is shown that $R_{P_{p_0},\mathcal{K}}$ is bounded on L^2 if and only if every vertex of the Newton diagram of P_{p_0} has at least one even component. It is straightforward to check that the same conclusion holds for the truncated operators $R_{P_{p_0},\mathcal{K}_r}$.

This completes the proof of Theorem 3.1.

8. The conclusion of the proof of Theorem 1.2

Consider the following truncations of the multiparameter singular Radon transform $R_{P,K}$ (from the statement of Theorem 1.2):

$$R_{P,K_r} f(x,y) \; = \iint_{\mathbb{R}^2} f(x-t, y-P(s,t))\, K_r(s,t) ds dt$$

where

$$K_r(s,t) \; = \; \sum_{I \in \mathcal{F}_2^{0,r}} \phi_I^{(I)}(s,t)$$

is a truncation of the product kernel K. The multiplier $M_r(\eta,\lambda)$ of R_{P,K_r} is precisely equal to the multiplier of T_r; that is,

$$M_r(\lambda,\eta) \; = \; \sum_{I \in \mathcal{F}_2^{0,r}} \iint_{\mathbb{R}^2} e^{2\pi i(\eta t + \lambda P(s,t))} \phi_I^{(I)}(s,t)\, ds dt.$$

Thus the uniform $L^2(\mathbb{H}^1)$ boundedness of $T_{P,\mathcal{F}}$ (where we seek uniformity over $P \in \mathcal{V}_\Delta$ and the truncations $\mathcal{F}$) is equivalent to the uniform $L^2(\mathbb{R}^2)$ boundedness of R_{P,K_r} where uniformity in r is also required. This is the main statement in Theorem 1.2. When $K(s,t) = \mathcal{K}(s,t) = 1/st$ is the double Hilbert transform kernel, we can apply Theorem 5.1 from [12] exactly as we did for the Ricci-Stein theorem from the Introduction to conclude that

$$\sup_r \sup_{P \in \mathcal{V}_\Delta} \|R_{P,\mathcal{K}_r}\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \; < \; \infty$$

if and only if every $\alpha = (\alpha_1, \alpha_2) \in \Delta$ has at least one even component. The *only if* part of the statement is an easy computation of the multiplier $M_r(\lambda,\eta)$ associated to a single monomial $P(s,t) = s^j t^k$ where both j and k are odd (see [5]).

This completes the proof of Theorem 1.2.

Appendix A. Proof of (19)

In this appendix we give a proof of the oscillatory integral estimate (19). Recall

$$\mathcal{I}_j \;=\; \int_{\mathbb{R}} e^{2\pi i Q(s)}\, 2^{-j}\Phi(2^{-j}s)\, ds$$

where Q is either

$$\lambda\Big[ys^{p_0} + \sum_{p\geq 1} c_{p,0}s^p + \sum_{1\leq p<p_0} \psi_p(y-t)s^p\Big]$$

or

$$\lambda\Big[\sum_{p\geq 1} c_{p,0}s^p + \sum_{1\leq p<p_0} \psi_p(y-t)s^p\Big].$$

For ease in notation, we will assume Q is the latter. When considering the former instead, without loss of generality one may assume there exists an $c_{p,0}\neq 0$ for some $p>p_0$; otherwise, we would be in the polynomial case where we can appeal to (17).

Let $p_n<\cdots<p_1$ enumerate the values of $1\leq p<p_0$ such that $\psi_p\not\equiv 0$. In this case, for each $1\leq r\leq n$, there is an $\ell_r\geq 1$ such that $|\psi_{p_r}(y-t)|\sim 2^{\ell_r k}$ whenever $|y-t|\sim 2^k$. Hence $h_r := \lambda(c_{p_r,0}+\psi_{p_r}(y-t))$ satisfies $|h_r|\gtrsim |\lambda|2^{\ell_r k}$ whenever $|y-t|\sim 2^k \ll 1$ and with this notation, (19) reads

$$|\mathcal{I}_j| \;\lesssim_P\; \big[|h_{p_r}|2^{pj}\big]^{-\epsilon_r} \;\lesssim\; \big[|\lambda|2^{\ell_r k}2^{p_r j}\big]^{-\epsilon_r} \tag{27}$$

for every $1\leq r\leq p_0-1$ and for some $\epsilon_r>0$.

We fix an $1\leq L<p_0$ and establish (27) with $r=L$. First of all, we have $|\psi_{p_r}(y-t)s^{p_r}|\sim 2^{\ell_r k}2^{p_r j}$ and thus let us name these quantities $\theta_r(k,j) := 2^{\ell_r k}2^{p_r j}$; they will be used to control the contribution of each term of Q to some derivative of Q itself.
We introduce a sequence of small parameters $0<\delta_1\ll\delta_2\ll\cdots\ll\delta_{L-1}\ll 1$ depending on P, which will be chosen later, and define for each $1\leq r\leq L$ sets

$$U_r := \Big\{ j \;:\; \theta_1(k,j) < \delta_1\theta_L(k,j),\\ \vdots \\ \theta_{r-1}(k,j) < \delta_{r-1}\theta_L(k,j),\\ \text{and}\\ \theta_r(k,j) \geq \delta_r\theta_L(k,j)\Big\}.$$

Notice that for U_1 the first conditions are vacuous and we only stipulate $\theta_1(k,j)\geq \delta_1\theta_L(k,j)$, and for U_L the last condition is vacuous and we only stipulate $\theta_s(k,j)<\delta_s\theta_L(k,j)$ for all $s=1,\dots,L-1$. It is immediate to see that these sets form a partition of the set of all possible j's.

Suppose that $j\in U_r$ for some $1\leq r\leq L$. We examine the p_r-th derivative of Q:

$$Q^{(p_r)}(s) = \lambda\Big[\sum_{i=1}^{r} \frac{p_i!}{(p_i-p_r)!}\psi_{p_i}(y-t)s^{p_i-p_r} + \frac{p_*!}{(p_*-p_r)!}c_* s^{p_*-p_r} + O(s^{p_*-p_r+1})\Big]$$

where $p_* \geq p_r$ is the first exponent such that $c_{p_*,0} \neq 0$. Noting $|s| \sim 2^j \ll 1$ and $j \in U_r$, the contribution of the mixed terms with $i < r$ is at most

$$C_P \sum_{i<r} \theta_i(k,j) 2^{-p_r j} \leq C_P(\delta_1 + \ldots + \delta_{r-1}) \theta_L(k,j) 2^{-p_r j},$$

while the contribution of the mixed term with $i = r$ is $\sim \theta_r(k,j)2^{-p_r j} > \delta_r \theta_L(k,j)2^{-p_r j}$. By choosing the constants δ_i to be sufficiently small (depending on P) and decreasing fast enough we have then

$$\Big| \sum_{i=1}^{r} \frac{p_i!}{(p_i - p_r)!} \psi_{p_i}(y-t) s^{p_i - p_r} \Big| \gtrsim \theta_L(k,j) 2^{-p_r j}$$

when $j \in U_r$.

As for the contribution of the remaining terms, we have $|c_*| s^{p_* - p_r} \sim 2^{(p_* - p_r)j}$. If $\theta_L(k,j) \not\sim 2^{p_* j}$ we have then $|Q^{(p_r)}(s)| \gtrsim |\lambda| \theta_L(k,j) 2^{-p_r j}$, implying that

$$|\mathcal{I}_j| \lesssim \big(2^{p_r j} |\lambda| \theta_L(k,j) 2^{-p_r j}\big)^{-1/p_r} = \big(|\lambda| \theta_L(k,j)\big)^{-1/p_r}$$

by van der Corput's lemma. Hence (27) holds in this case.
Otherwise, in the case $2^{p_* j} \sim \theta_L(k,j)$, we have the bound $|Q^{(p_*)}(s)| \gtrsim 1$ since every $2^{\ell_r k} \ll 1$. Another application of van der Corput's lemma shows

$$|\mathcal{I}_j| \lesssim \big(2^{p_* j} |\lambda|\big)^{-1/p_*} \sim \big(|\lambda| \theta_L(k,j)\big)^{-1/p_*}.$$

This completes the proof of (27).

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Marco Vitturi: School of Mathematical Sciences, University College Cork, Western Gateway Building, Western Road, Cork T12 XF62 , Ireland

Email address: marco.vitturi@ucc.ie

James Wright: School of Mathematics and Maxwell Institute for Mathematical Sciences, University of Edinburgh, JCMB, King's Buildings, Peter Guthrie Tait Road, Edinburgh EH9 3FD, Scotland

Email address: J.R.Wright@ed.ac.uk